NON CYCLIC FUNCTIONS IN THE HARDY SPACE OF THE BIDISC WITH ARBITRARY DECREASE

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Abstract. We construct an example to show that no condition of slow decrease of the modulus of a function is sufficient to make it cyclic in the Hardy space of the bidisc. This is similar to what is well known in the case of the Hardy space of the disc, but in contrast to the case of the Bergman space of the disc.

1. Background

A cyclic vector f for a given operator ϕ from a topological vector space X to itself is one such that $\operatorname{Span}\{\phi^n(f), n \in \mathbb{Z}_+\}$ is dense in X. In the classical framework of the Hardy space $H^2(\mathbb{D})$ on the unit disk with the shift operator $\phi(f)(z) := zf(z)$, a cyclic function is one that verifies that $\mathbb{C}[Z]f$ is dense in $H^2(\mathbb{D})$, where $\mathbb{C}[Z]$ denotes the set of holomorphic polynomials. The characterization of those as outer functions goes back to Beurling, see [Ga] or [Ko] for details. For any $f \in H^2(\mathbb{D})$,

$$f(\zeta) := B(\zeta) \exp\left(\int_0^{2\pi} \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta} \ (g(\theta)d\theta - d\mu(\theta)) \right),$$

where B is a Blaschke product, $e^g \in L^2(\partial\mathbb{D})$ and μ is a positive singular measure. The factor containing only μ is called a singular inner function. The function f is called *outer* when $\mu = 0$ and $B = 1$ (or, equivalently, $\log|f|$ is the Poisson integral of its boundary values).

We are interested in the situation in $H^2(\mathbb{D}^2)$, where $\mathbb{D}^2 := \mathbb{D}\times\mathbb{D} \subset \mathbb{C}^2$, and

$$H^2(\mathbb{D}^2) := \left\{ f \in \operatorname{Hol}(\mathbb{D}^2) : \|f\|_2^2 := \sup_{r<1} \int_0^{2\pi}\int_0^{2\pi} |f(re^{i\theta_1}, re^{i\theta_2})|^2 d\theta_1 d\theta_2 < \infty \right\}.$$

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A *cyclic* function is $f \in H^2(\mathbb{D}^2)$ such that $\mathbb{C}[Z_1, Z_2]f$ is dense in $H^2(\mathbb{D}^2)$.

Rudin [Ru] proved that the analogous equivalence between being outer (in an appropriate sense) and cyclic fails in the case of the polydisk. A lot of work on analogous problems was later carried out by Hedenmalm, see for instance [He].

Since functions with the same modulus are cyclic or not simultaneously — because if f_1 is cyclic and $|f_2(z)| \geq |f_1(z)|$, for any $z \in \mathbb{D}^2$, then f_2 is cyclic — it is natural to look for conditions on the size of f that would be necessary or sufficient for cyclicity. In general, $|f|$ cannot be allowed to vanish, nor decrease too fast near the boundary, in order for f to be cyclic.

One type of necessary condition can be obtained by restricting functions to the diagonal. Rudin [Ru, p. 53] noticed that the map $f \mapsto Rf(\zeta) := f(\zeta, \zeta)$ is bounded and onto from $H^2(\mathbb{D}^2)$ to the Bergman space $A^2(\mathbb{D}) := \mathrm{Hol}(\mathbb{D}) \cap L^2(\mathbb{D})$ (this is a quite general phenomenon, see [HoOb] for instance). Thus if f is cyclic in $H^2(\mathbb{D}^2)$, then Rf is cyclic in $A^2(\mathbb{D})$, that is to say, $\mathbb{C}[Z]Rf$ is dense in $A^2(\mathbb{D})$.

Borichev [Bo] proved that if $f \in A^2(\mathbb{D})$ and $|f|$ decreases slowly enough, then f is cyclic in $A^2(\mathbb{D})$. More precisely, we call *weight function* a non increasing function v from $(0, 1]$ to $(0, \infty)$ such that $\lim_{t\to 0} v(t) = \infty$ and $v(t^2) \leq Cv(t)$ for some $C > 0$. The following is a special case of [Bo, Theorem 2]:

Theorem 1.1. *If $|f(\zeta)| \geq \exp(-v(1 - |\zeta|))$ for a weight function v such that*

$$\int_0 \frac{v(t)^2}{t(\ln t)^2} dt < \infty,$$

then f is cyclic in $A^2(\mathbb{D})$.

It has been known for a long time that no sufficient condition of this type can hold for $H^2(\mathbb{D})$; indeed, no non-trivial singular inner function may be cyclic, but one can construct them with arbitrarily slow decrease. On the other hand, in view of the sufficient condition in the Bergman space case, one may wonder whether a similar sufficient condition could hold for $H^2(\mathbb{D}^2)$. The purpose of this note is to show that it is not the case, as the example below shows.

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2. The example

For $(z_1, z_2) \in \mathbb{D}^2$, let $\delta(z) := \max(1-|z_1|, 1-|z_2|)$ (this is comparable to the distance to the distinguished boundary of the bidisk $(\partial\mathbb{D})^2$).

Proposition 2.1. *For any weight function v, there exists $f \in H^2(\mathbb{D}^2)$ such that $\log|f(z)| \geq -v(\delta(z))$, but f is not cyclic for $H^2(\mathbb{D}^2)$.*

Proof. Let μ_v be a singular positive measure on $\partial\mathbb{D}$ such that $\mu_v(I) \leq c_0|I|v(|I|)$, for c_0 small enough, then the singular function

$$f_0(\zeta) := \exp\left(-\int_0^{2\pi} \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\, d\mu_v(\theta)\right)$$

is not cyclic for $H^2(\mathbb{D})$ and verifies $\log|f_0(z)| \geq -v(1-|z|)$.

Now set $f(z) := f_0(z_1 z_2)$. Since

$$\max(1-|z_1|, 1-|z_2|) \leq 1-|z_1 z_2| \leq 1-|z_1|+1-|z_2|,$$

an appropriate modification of the constant c_0 yields $\log|f(z)| \geq -v(\delta(z))$.

If $f_0(\zeta) = \sum_k \alpha_k \zeta^k$, then $f(z) = \sum_k \alpha_k z_1^k z_2^k = \sum_{j,k} a_{j,k} z_1^j z_2^k$, with $\sum_{j,k}|a_{j,k}|^2 = \sum_k |\alpha_k|^2 < \infty$, so $f \in H^2(\mathbb{D}^2)$ (or in this case we just might notice we have bounded functions).

In what follows, we freely use the fact that any function in $H^2(\mathbb{D})$ or $H^2(\mathbb{D}^2)$ admits almost everywhere defined non-tangential boundary values, and that its norm is equal to the L^2 norm on the circle (resp. torus).

Suppose there exists a sequence of polynomials $P_n(z_1, z_2)$ such that

$$\lim_n \int_0^{2\pi}\int_0^{2\pi} \left|P_n(e^{i\theta_1}, e^{i\theta_2}) f(e^{i\theta_1}, e^{i\theta_2}) - 1\right|^2 d\theta_1 d\theta_2 = 0.$$

Using the change of variables $\theta_1 = \theta$, $\theta_2 = \theta+\alpha$, we get

$$\lim_n \int_0^{2\pi}\left(\int_0^{2\pi} \left|P_n(e^{i\theta}, e^{i(\theta+\alpha)}) f_0(e^{i(2\theta+\alpha)}) - 1\right|^2 d\theta\right) d\alpha = 0,$$

which implies that there exists an increasing sequence n_k of integers such that for almost any α,

$$\lim_k \int_0^{2\pi} \left|P_{n_k}(e^{i\theta}, e^{i(\theta+\alpha)}) f_0(e^{i(2\theta+\alpha)}) - 1\right|^2 d\theta = 0.$$

For such an α, let $g_\alpha(\zeta) := f_0(e^{i\alpha}\zeta^2)$. This is again a singular function, but we have a sequence of polynomials $q_k(\zeta) := P_{n_k}(\zeta, e^{i\alpha}\zeta)$ such that $\|q_k g_\alpha - 1\|_{H^2(\mathbb{D})} \to 0$, which is a contradiction.

□

References

[Bo] Alexander Borichev, Estimates from below and cyclicity in Bergman-Type Spaces, Internat. Math. Res. Notices, 1996, no. 12, 603–611.

[Ga] John B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

[He] Håkan Hedenmalm, Outer functions in function algebras of the bidisc, Trans. Amer. Math. Soc. **306** (1988), no. 2, 697–714.

[HoOb] Charles Horowitz, Daniel M. Oberlin, Restriction of H^p functions to the diagonal of U^n, Indiana Univ. Math. J. **24** (1974/75), 767–772.

[Ko] Paul Koosis, *Introduction to H_p spaces*, London Math. Soc. Lecture Notes **40**, 1980.

[Ru] Walter Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969.

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